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A Series Useful in the Study of Difference Equations*

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The series treated in this paper had its origin in the study of difference equations with varying difference interval. It is hoped to follow in a short time with a paper on this subject.

PART I

We shall be concerned with series of the type

$$\sum_{n=1}^{\infty} C_n[(z + h_1(z))(z + h_2(z)) \cdots (z + h_n(z))]^{-1}. \tag{1}$$

Here $z = x + yi$ with x and y real. We shall require in addition that $0 < \epsilon \leqslant h_n(z) \leqslant E(n)$, that $h_n(z) \to \infty$ when $n \to \infty$ and that $x + h_n(z)$ increase with x.

1. CONVERGENCE THEOREMS

We now give a lemma which is well known. However, the proof is so short that it is given as adding to the understanding of the work which follows.

Let $\Delta b_j(z) = b_{j+1}(z) - b_j(z)$:

LEMMA 1. *Hypotheses:* (1) *$a_n(z)$ and $b_n(z)$, for all n, are defined at all points of a region, R, of the complex plane.* (2) *$\sum_{n=1}^{\infty} a_n(z)$ converges uniformly over R.* (3) *$b_n(z)$ is uniformly bounded over R.* (4) *$\sum_{j=m}^{m'} | \Delta b_j(z) |$ is uniformly bounded over R. Conclusion: $\sum_{n=1}^{\infty} a_n(z)\, b_n(z)$ converges uniformly over R.*

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PROOF. Using summation by parts

$$\sum_{n=m}^{m'} a_n(z)\, b_n(z) = b_n(z) \sum_{j=m}^{n-1} a_j \Big]_m^{m'+1} - \sum_{n=m}^{m'} \Delta_n b_n(z) \sum_{j=m}^{n} a_j$$

$$\left| \sum_{n=m}^{m'} a_n(z)\, b_n(z) \right| \leqslant | b_{m'+1}(z) | \left| \sum_{j=m}^{m'} a_j \right| + \sum_{n=m}^{m'} \Big| | \Delta_n b_n(z) | \Big| \left| \sum_{j=m}^{n} a_j \right|$$

$$\leqslant M\epsilon + N\epsilon < \eta$$

which proves the lemma.

Let

$$b_n(z) = \frac{(z_0 + h_1(z_0))\,(z_0 + h_2(z_0)) \cdots (z_0 + h_n(z_0))}{(z + h_1(z))\,(z + h_2(z)) \cdots (z + h_n(z))}. \tag{2}$$

This expression plays an important role in the discussions that follow.

Let

$$b_n(x) = \frac{(x_0 + h_1(z_0))\,(x_0 + h_2(z_0)) \cdots (x_0 + h_n(z_0))}{(x + h_1(z))\,(x + h_2(z)) \cdots (x + h_n(z))}. \tag{3}$$

THEOREM 1. *If $\sum_{n=1}^{\infty} [h_n(z_0)]^{-2}$ converges and $0 \leqslant x_0 \leqslant x$ then given any positive numbers, a and b, and a positive integer, p, there exists an N such that*

$$| b_n(z) | < \frac{N}{a(x - x_0)^p + b}, \qquad n > p. \tag{4}$$

PROOF:

$$| b_n(z) | = \frac{| z_0 + h_1(z_0) | \cdot | z_0 + h_2(z_0) | \cdots | z_0 + h_n(z_0)}{(x_0 + h_1(z_0))\,(x_0 + h_2(z_0)) \cdots (x_0 + h_n(z_0))}$$

$$\cdot \frac{(x + h_1(z))\,(x + h_2(z)) \cdots (x + h_n(z))}{| z + h_1(z) | \cdot | z + h_2(z) | \cdots | z + h_n(z) |} \cdot b_n(x)$$

$$\leqslant \left\{ \left[1 + \left(\frac{y_0}{x_0 + h_1(z_0)}\right)^2\right] \left[1 + \left(\frac{y_0}{x_0 + h_2(z_0)}\right)^2\right] \right.$$

$$\left. \cdots \left[1 + \left(\frac{y_0}{x_0 + h_n(z_0)}\right)^2\right] \right\}^{1/2} b_n(x) < \bar{N} b_n(x). \tag{5}$$

This follows from the fact that

$$\frac{(x + h_1(z)) \cdots (x + h_n(z))}{| z + h_1(z) | \cdots | z + h_n(z) |} \leqslant 1,$$

and the fact that $\sum_{n=1}^{\infty} [h_n(z_0)]^{-2}$ converges.

Hence

$$\left| b_n(z) \leqslant \frac{\bar{N}(x_0 + h_1(z_0)) \cdots (x_0 + h_p(z_0))}{(x + h_1(z)) \cdots (x + h_p(z))}\right.$$

since

$$\frac{(x_0 + h_{p+1}(z_0)) \cdots (x_0 + h_n(z_0))}{(x + h_{p+1}(z)) \cdots (x + h_n(z))} \leqslant 1.$$

Consequently

$$| b_n(z) | \leqslant \frac{C\bar{N}(x_0 + h_1(z_0)) \cdots (x_0 + h_p(z_0))}{C[x^p + h_1(z)\, h_2(z) \cdots h_p(z)]} < \frac{N}{a(x - x_0)^p + b}$$

provided

$$N > C\bar{N}(x_0 + h_1(z_0)) \cdots (x_0 + h_p(x_0)), \qquad c > a$$

and

$$Ch_1(z)\, h_2(z) \cdots h_p(z) > b.$$

It is always possible so to choose C since p is fixed and $h_n(z) \geqslant \epsilon > 0$.

We have made the hypothesis that $n > p$ which is trivial in convergence problems.

THEOREM II. *$b_n(z)$ is uniformly bounded in the half plane, $x \geqslant 0$.*

This was proved incidentally in the proof of the last theorem since $0 < b_n(x) \leqslant 1$.

LEMMA II.

$$\frac{| z - z_0 + h_{n+1}(z) - h_{n+1}(z_0) |}{x - x_0 + h_{n+1}(z) - h_{n+1}(z_0)} \leqslant \frac{| z - z_0 |}{x - x_0}, \qquad x > x_0 .$$

PROOF.

$$\frac{| z - z_0 + h_{n+1}(z) - h_{n+1}(z_0) |}{x - x_0 + h_{n+1}(z) - h_{n+1}(z_0)}$$

$$= \left[1 + \frac{[(y - y_0)/(x - x_0)]^2}{\{1 + [h_{n+1}(z) - h_{n+1}(z_0)]/(x - x_0)\}^2}\right]^{1/2} \leqslant \left[1 + \left(\frac{y - y_0}{x - x_0}\right)^2\right]^{1/2}$$

$$= \frac{| z - z_0 |}{x - x_0}. \tag{6}$$

THEOREM III. *If $x \geqslant x_0 \geqslant 0$, and $h_{n+1}(z) - h_n(z_0) \geqslant 0$ and if $\sum_{n=1}^{\infty} (1/h_n(z))^2$ converges uniformly when $x \geqslant x_0$, then $\sum_{n=m}^{m'} | \Delta b_n(z) |$ is uniformly bounded over the portion of the complex plane determined by*

$$\frac{y - y_0}{x - x_0} \leqslant a(x - x_0)^p + b.$$

Here a and b are any positive constants and p any positive integer, $x > x_0$.

PROOF. From the definition

$$|\Delta_n b_n(z)| = \frac{|z - z_0 + h_{n+1}(z) - h_{n+1}(z_0)|}{|z + h_{n+1}(z)|} |b_n(z)|.$$

But by Theorem I

$$|b_n(z)| \leqslant N b_n(x).$$

Hence, using the previous lemma,

$$|\Delta b_n(z)| \leqslant N \frac{|z - z_0 + h_{n+1}(z) - h_{n+1}(z_0)|}{x - x_0 + h_{n+1}(z) - h_{n+1}(z_0)} \cdot \frac{x + h_{n+1}(z)}{|z + h_{n+1}(z)|}$$

$$\cdot \frac{x - x_0 + h_{n+1}(z) - h_{n+1}(z_0)}{x + h_{n+1}(z)} b_n(x)$$

$$\leqslant -N \frac{|z - z_0|}{x - x_0} \Delta b_n(x).$$

From this

$$\sum_{n=m}^{m'} |\Delta b_n(z)| \leqslant -N \sqrt{1 + \left(\frac{y - y_0}{x - x_0}\right)^2} (b_{m'+1}(x) - b_m(x))$$

$$\leqslant N \sqrt{1 + \left(\frac{y - y_0}{x - x_0}\right)^2} b_m(x)$$

$$\leqslant M \frac{\sqrt{1 + [(y - y_0)/(x - x_0)]^2}}{a(x - x_0)^p + b}$$

$$\leqslant M \frac{\sqrt{1 + [a(x - x_0)^p + b]^2}}{a(x - x_0)^p + b} < \bar{M}$$

which completes the proof.

LEMMA III. *If* $z_0 = 0$, $x > 0$ *and*

$$\frac{|y|}{x + h_n(z) - h_n(z_0)} < m$$

then

$$\frac{|z + h_n(z) - h_n(z_0)|}{|z + h_n(z)| - h_n(0)} < M,$$

where m *and* M *are constants.*

PROOF. We note that $|z + h_n(z)| - h_n(0) > 0$. This follows from the fact that $x + h_n(z) > 0 + h_n(0)$. This is true since $x + h_n(z)$ increases with x.

$$| z + h_n(z) | - h_n(0) \geqslant x + h_n(z) - h_n(0).$$

Hence

$$\frac{| z + h_n(z) - h_n(0) |}{| z + h_n(z) | - h_n(0)} \leqslant \frac{| z + h_n(z) - h_n(0) |}{x + h_n(z) - h_n(0)}$$

$$= \left\{ 1 + \left(\frac{y}{x + h_n(z) - h_n(0)} \right)^2 \right\}^{1/2} < \sqrt{1 + m^2} < M.$$

The conditions just given, namely, $z_0 = 0$, $x > 0$,

$$\frac{| y |}{x + h_n(z) - h_n(0)} \leqslant m,$$

serve to define a region, S, as follows. For example if $| h_n(z) - h_n(0) |$ has an upper bound g then a sectorial region is defined by $| y | \leqslant m(x + g)$. If $| y | \leqslant m(x + h_n(z) - h_n(0))$ then z lies in S.

THEOREM IV. *$\sum_{n=m}^{m'} | \Delta b_n(z) |$ is uniformly bounded over the region, S, defined above.*

PROOF:

$$| \Delta b_n(z) | = \frac{| z + h_n(z) - h_n(0) |}{| z + h_n(z) | - | h_n(0) |} \Delta | b_n(z) | .$$

Now $| b_n(z) | \leqslant N$ and by Lemma III

$$\frac{| z + h_n(z) - h_n(0) |}{| z + h_n(z) | - | h_n(0) |} < M.$$

The theorem follows by summation.

THEOREM V. *If series (1) converges at z_0 and if $x_0 \geqslant 0$ and if $\sum_{n=1}^{\infty} (1/h_n(z))^2$ converges uniformly over a sectorial region, R, determined by*

$$| y - y_0 | (x - x_0)^{-1} \leqslant a(x - x_0)^p + b;$$

then (1) converges uniformly over R also.

PROOF. This theorem is an immediate consequence of Lemma I and Theorems II and III.

THEOREM VI. *If (1) converges at* 0 *then it converges uniformly over the region, S, of Lemma III.*

PROOF. The proof again is an immediate consequence of Lemma I and Theorem IV.

THEOREM VII. *If (1) converges absolutely at* z_0, *when* $x_0 \geqslant 0$ *then it converges absolutely uniformly over the half-plane determined by* $x \geqslant x_0$.

PROOF. This theorem follows from the fact that $z + h_n(z)$ is increasing in x.

THEOREM VIII. *If (1) converges at* $z = z_0$, $x_0 \geqslant 0$ *then it converges at all points of the half plane* $x > x_0$.

The conditions on h in Theorems V, VI, VII, and VIII are those in the lemmas and theorems referred to.

2. UNIQUENESS THEOREMS

THEOREM IX. *If*

$$\sum_{n=1}^{\infty} \frac{c_n}{(z + h_1(z))(z + h_2(z)) \cdots (z + h_n(z))} \equiv 0 \tag{7}$$

over its half-plane of convergence, $x \geqslant x_0 \geqslant 0$, *then* $c_n = 0$, $n = 1, 2, \cdots$.

PROOF. When x becomes infinite each term approaches zero. Convergence is uniform. We can assume the c's functions of $h(z)$ but add the hypothesis, c_n/z approaches zero when x becomes infinite.

$$\frac{c_1}{z + h_1(z)} + \frac{c_2}{(z + h_1(z))(z + h_2(z))} + \frac{c_3}{(z + h_1(z))(z + h_2(z))(z + h_3(z))} + \cdots \equiv 0 \tag{8}$$

from which

$$c_1 + \frac{c_2}{z + h_2(z)} + \frac{c_3}{(z + h_2(z))(z + h_3(z))} + \cdots \equiv 0. \tag{9}$$

This is a series of the same type as (8). It converges uniformly $x \geqslant x_0$. Let x become infinite and we have $c_1 = 0$. We now begin all over again in a familiar way and get $c_2 = 0$ and then $c_3, c_4, \cdots c_n, \cdots$.

THEOREM X. *$\Omega(z)\,\bar{\Omega}(z) = 0$ at an infinite number of points in the finite half plane $x_0 \leq x < M$ then $\Omega(z)$ and/or $\bar{\Omega}(z)$ is identically zero.*

PROOF. Either $\Omega(z)$ or $\bar{\Omega}(z)$ must have an infinite number of finite zeros. This is impossible for an analytic function unless it is identically zero.

PART II

In this part of the paper the independent variable is real. It is denoted by x. We call attention to the fact that $h_n(x)$ is real and positive and that $h_{n+1}(x) > h_n(x)$ and $x + h_n(x)$ increasing with x also that $x + h_n(x) > 0$.

3. STEP-UP THEOREMS

We consider series (1) and assume that it is absolutely convergent for all values of x considered.

Let

$$| c_n | = a_n$$

and let

$$\Omega_1(x) = \sum_{j=1}^{\infty} \frac{c_j}{(x + h_1(x))\,(x + h_2(x)) \cdots (x + h_j(x))} \tag{10}$$

$$\Omega_2(x) = \sum_{j=1}^{\infty} \frac{a_j}{(x + h_1(x))\,(x + h_2(x)) \cdots (x + h_j(x))}. \tag{11}$$

We shall carry on certain operations on series (10) and (11). These operations are to be precisely set forth in the theorems.

THEOREM XI. *If $\Sigma_{n=1}^{\infty} 1/h_n(x)$ diverges then*

$$\Omega_1(x) = \sum_{n=1}^{\infty} \frac{{}_2\varphi_n}{(x + h_2(x))\,(x + h_3(x)) \cdots (x + h_{n+1}(x))} \tag{12}$$

where

$$\begin{aligned} {}_2\varphi_n = c_n &+ c_{n-1}(h_n(x) - h_1(x)) + c_{n-2}(h_n(x) - h_1(x))\,(h_{n-1}(x) - h_1(x)) + \cdots \\ &+ c_1(h_n(x) - h_1(x))\,(h_{n-1}(x) - h_1(x)) \cdots (h_2(x) - h_1(x)). \end{aligned}$$

PROOF. Let

$$\frac{1}{x+h_1(x)} = \frac{1}{x+h_2(x)} + \epsilon_1 ,$$

where

$$\epsilon_1 = \frac{h_2(x) - h_1(x)}{(x + h_1(x))\,(x + h_2(x))};$$

$$\epsilon_1 = \frac{h_2(x) - h_1(x)}{(x + h_2(x))\,(x + h_3(x))} + \epsilon_2 ,$$

where

$$\epsilon_2 = \frac{(h_2(x) - h_1(x))\,(h_3(x) - h_1(x))}{(x + h_1(x))\,(x + h_2(x))\,(x + h_3(x))} .$$

In general with the same notation we have

$$\epsilon_n = \frac{(h_2(x) - h_1(x))\,(h_3(x) - h_1(x)) \cdots (h_{n+1}(x) - h_1(x))}{(x + h_1(x))\,(x + h_2(x)) \cdots (x + h_{n+1}(x))} .$$

From this

$$\epsilon_n = \frac{1}{x + h_1(x)} \, \frac{1}{\left\{\left(1 + \dfrac{x + h_1(x)}{h_2(x) - h_1(x)}\right)\left(1 + \dfrac{x + h_1(x)}{h_3(x) - h_1(x)}\right) \cdots \left(1 + \dfrac{x + h_1(x)}{h_{n+1}(x) - h_1(x)}\right)\right\}} .$$

This approaches zero since $\sum_1^\infty 1/h_n(x)$ diverges. Consequently we have

$$\frac{c_1}{x + h_1(x)} = \frac{c_1}{x + h_2(x)} + \frac{c_1(h_2(x) - h_1(x))}{(x + h_2(x))\,(x + h_3(x))} + \cdots$$
$$+ \frac{c_1(h_2(x) - h_1(x))\,(h_3(x) - h_1(x)) \cdots (h_n(x) - h_1(x))}{(x + h_2(x))\,(x + h_3(x)) \cdots (x + h_{n+1}(x))} + \cdots .$$

In precisely the same way we find

$$\frac{c_2}{(x + h_1(x))\,(x + h_2(x))} = \frac{c_2}{(x + h_2(x))\,(x + h_3(x))}$$
$$+ \frac{c_2(h_3(x) - h_1(x))}{(x + h_2(x))\,(x + h_3(x))\,(x + h_4(x))} + \cdots$$
$$+ \frac{c_2(h_3(x) - h_1(x))\,(h_4(x) - h_1(x)) \cdots (h_{n-1}(x) - h_1(x))}{(x + h_2(x))\,(x + h_3(x)) \cdots (x + h_n(x))} + \cdots .$$

Proceeding in this manner

$$\frac{c_j}{(x+h_1(x))\,(x+h_2(x))\cdots(x+h_j(x))}$$

$$= \frac{c_j}{(x+h_2(x))\,(x+h_3(x))\cdots(x+h_{j+1}(x))}$$

$$+ \frac{c_j(h_{j+1}(x)-h_1(x))}{(x+h_2(x))\,(x+h_3(x))\cdots(x+h_{j+2}(x))}$$

$$+ \frac{c_j(h_{j+1}(x)-h_1(x))\,(h_{j+2}(x)-h_1(x))}{(x+h_2(x))\,(x+h_3(x))\cdots(x+h_{j+3}(x))}+\cdots.$$

These series considered together form a double series which, summed by rows, produces $\Omega_1(x)$. This double series is absolutely convergent. To see this, do to series (11) for $\Omega_2(x)$ precisely what we have just done to series (10) for $\Omega_1(x)$. The resulting double series is of all positive terms and can be summed to $\Omega_2(z)$ by rows. It consequently is convergent. This double series is a majorant series for the series just obtained for $\Omega_1(x)$, which, consequently, is absolutely convergent and can be summed by columns. We thus obtained the result of the theorem.

We repeat this same process with series (12). We find

$$\Omega_1(x) = \sum_{n=1}^{\infty} \frac{{}_3\varphi_n}{(x+h_3(x))\,(x+h_4(x))\cdots(x+h_{n+2}(x))}.$$

Here

$${}_3\varphi_n = {}_2\varphi_n + {}_2\varphi_{n-1}(h_{n+1}(x)-h_2(x)) + {}_2\varphi_{n-2}(h_{n+1}(x)-h_2(x))\,(h_n(x)-h_2(x))$$
$$+\cdots+{}_2\varphi_1(h_{n+1}(x)-h_2(x))\,(h_n(x)-b_2(x))\cdots(h_3(x)-h_2(x)).$$

Proceeding in this manner we have the following theorem.

THEOREM XII.

$$\Omega_1(z) = \sum_{n=1}^{\infty} \frac{{}_k\varphi_n}{(x+h_k(x))\,(x+h_{k+1}(x))\cdots(x+h_{n+k-1}(x))},$$

where

$${}_k\varphi_n = {}_{k-1}\varphi_n + {}_{k-1}\varphi_{n-1}(h_{n+k-2}(x)-h_{k-1}(x))$$
$$+ {}_{k-1}\varphi_1(h_{n+k-2}(x)-h_{k-1}(x))\cdots(h_k(x)-h_{k-1}(x)).$$

4. Step-Down Theorem

Theorem XIII. *If*

$$Z(x) = \frac{a_1}{x + h_p(x)} + \frac{a_2}{(x + h_p(x)\,(x + h_{p+1}(x))} + \cdots + \frac{a_n}{(x + h_p(x)\,(x + h_{p+1}(x)) \cdots (x + h_{p+n-1}(x))} + \cdots$$

then

$$Z(x) = \frac{a_1}{x + h_{p-1}(x)} + \frac{a_1(h_{p-1}(x) - h_p(x)) + a_2}{(x + h_{p-1}(x))\,(x + h_p(x))} + \cdots + \frac{a_{n-1}(h_{p-1}(x) - h_{p+n-1}(x)) + a_n}{(x + h_{p-1}(x))\,(x + h_p(x)) \cdots (x + h_{p+n-2}(x))} + \cdots .$$

Proof.

$$\frac{1}{x + h_p(x)} = \frac{1}{x + h_{p-1}(x)} + \epsilon \qquad \epsilon = \frac{h_{p-1}(x) - h_p(x)}{(x + h_p(x))\,(x + h_{p-1}(x))}$$

$$\frac{1}{(x + h_p(x))} = \frac{1}{x + h_{p-1}(x)} + \frac{h_{p-1}(x) - h_p(x)}{(x + h_p(x))\,(x + h_{p-1}(x))}$$

$$\frac{1}{(x + h_p(x))\,(x + h_{p+1}(x))} = \frac{1}{(x + h_{p-1}(x))\,(x + h_p(x))} + \frac{h_{p-1}(x) - h_{p+1}(x)}{(x + h_{p-1}(x))\,(x + h_p(x))\,(x + h_{p+1}(x))} .$$

. . . .

This step-down process can be continued as many times as desired.

Theorem XIV. *If $\rho(x)$ is any bounded positive function we can write*

$$\Omega_1(z) = \sum_{j=1}^{\infty} \bar{c}_j \frac{A(h, \rho)}{(x + \rho(x) + h_1(x))\,(x + \rho(x) + h_2(x)) \cdots (x + \rho(x) + h_j(x))} .$$

Proof. Simply apply the ϵ process of Theorem XI.

5. Multiplication Theorem

Theorem XV. *Let*

$$\Omega(x) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{(x + h_1(x))\,(x + h_2(x)) \cdots (x + h_{n+1}(x))}$$

and

$$\bar{\Omega}(x) = \sum_{n=0}^{\infty} \frac{b_{n+1}}{(x + h_1(x))\,(x - h_2(x)) \cdots (x + h_{n+1}(x))}$$

then

$$\Omega(x)\,\bar{\Omega}(x) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{b_j\ {}_{j+1}\varphi_{n-j}}{(x + h_{n+1}) \cdots (x + h_{j+n+1})}\,. \tag{13}$$

Proof. By Theorem XI,

$$\Omega(x) = \sum_{n=0}^{\infty} a_{n+1} \frac{1}{(x + h_1(x)) \cdots (x + h_{n+1}(x))}$$

$$\Omega(x) = \sum_{n=0}^{\infty} {}_2\varphi_n \frac{1}{(x + h_2(x)) \cdot (x + h_{n+2}(x))}$$

$$\Omega(x) = \sum_{n=0}^{\infty} {}_3\varphi_n \frac{1}{(x + h_3(x)) \cdots (x + h_{n+3}(x))}$$

$$\vdots$$

$$\Omega(x) = \sum_{n=j}^{\infty} {}_j\varphi_n \frac{1}{(x + h_j(z)) \cdots (x + h_{j+n}(z))}\,.$$

We now multiply $\Omega(x)$ by each term of $\bar{\Omega}(x)$ as indicated and add

$$\frac{b_1}{x + h_1(x)} \sum_{n=0}^{\infty} {}_2\varphi_n \frac{1}{(x + h_2(x)) \cdots (x + h_{n+2}(x))}$$

$$\frac{b_2}{(x + h_1(x)\,(x + h_2(x)} \sum_{n=0}^{\infty} {}_3\varphi_n \frac{1}{(x + h_3(x)) \cdots (x + h_{n+3}(x))}$$

$$\frac{b_3}{(x + h_1(x))\,(x + h_2(x))\,(x + h_3(x))} \sum_{n=0}^{\infty} {}_4\varphi_n \frac{1}{(x + h_4(x)) \cdots (x + h_{n+4}(x))}$$

$$\vdots$$

$$\frac{b_j}{(x + h_1(x)) \cdots (x + h_j(x))} \sum_{n=0}^{\infty} {}_{j+1}\varphi_n \frac{1}{(x + h_{j+1}(x)) \cdots (x + h_{n+j+1}(x))}.$$

The double series which appears here can be summed by columns in as much as each row is absolutely convergent and it can be summed by rows. This follows from the fact that each of the series for $\Omega(z)$ converges absolutely. It is interesting to note that z occurs in the numerators of the series (13) only through h. Consequently if h is a constant we can write

$$\bar{\Omega}(x)\,\Omega(x) = \sum_{n=1}^{\infty} \frac{A_n}{(x + h_1)\,(x + h_2) \cdots (x + h_{n+1})}\,.$$

The A's are constant. If $h_n = n$ we have an ordinary factorial series, as is already known.

We are careful to note that the product converges uniformly over the common region of uniform convergence of the two factors.

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